Infinite coupling duals of N=2 gauge theories and new rank 1 superconformal field theories

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ABSTRACT: We show that a proposed duality [1] between infinitely coupled gauge theories and superconformal field theories (SCFTs) with weakly gauged flavor groups predicts the existence of new rank 1 SCFTs. These superconformal fixed point theories have the same Coulomb branch singularities as the rank 1 E_6, E_7, and E_8 SCFTs, but have smaller flavor symmetry algebras and different central charges. Gauging various subalgebras of the flavor algebras of these rank 1 SCFTs provides many examples of infinite-coupling dualities, satisfying an intricate set of consistency checks. They also provide examples of $N = 2$ conformal theories with marginal couplings but no weak-coupling limits.

A recently proposed expansion of the notion of S-duality [1] provides a new tool for exploring the set of $N = 2$ SCFTs by identifying them as factors in the infinite-coupling limits of Lagrangian field theories. The requirements of matching simple features of the low energy effective actions, global symmetry groups, and current algebra central charges, turn out to tightly constrain the possible properties of the SCFTs. In this paper we use this approach to map out rank 1 SCFTs (those with 1 complex-dimensional Coulomb branches) and find three new such theories.

The singular Seiberg-Witten curves describing the low energy effective actions on the Coulomb branches of rank 1 $N = 2$ SCFTs were found in [2, 3] and correspond to Kodaira's classification [4] of degenerations of one-dimensional families of elliptic curves. This gives a list of singularities associated to Lagrangian conformal theories, together with six strongly-coupled isolated fixed point singularities, three of which are characterized by having Coulomb branch vevs of dimensions 3, 4, and 6. Mass deformations of these singularities consistent with flavor symmetries E_6, E_7, and E_8 were constructed by Minahan and Nemeschansky [3], and their existence was deduced from string constructions [5].

We will argue on the basis of the proposed infinite coupling duality that there are other, inequivalent, mass deformations of these same singular curves, with correspondingly different flavor groups. In weakly-coupled examples, where we have a Lagrangian description, such inequivalent mass deformations are familiar: they correspond to different choices of gauge representations for the matter fields such that the gauge coupling remains marginal. Our interacting fixed-point examples with the same singularity but different mass deformations can, heuristically, be thought of in the same way: they are strongly coupled rank 1 gauge theories all with the same "gauge group" (corresponding to the singularity) but with different "matter content" (corresponding to the different mass deformations).

Because we have different deformations of the same Coulomb branch singularity, it is convenient to name the singularity by the dimensions of its Coulomb branch vevs instead of by its global symmetry algebra, which is a property of a particular deformation of the singularity. Thus SCFT[d_i] will denote a rank-n superconformal fixed point singularity with vevs of dimensions $d_1, \ldots, d_n$, and a particular deformation of this singularity with flavor symmetry algebra $\mathfrak{h}$ (corresponding to a particular SCFT) will be denoted by SCFT[d_i : $\mathfrak{h}$]. We denote a Lagrangian gauge theory with gauge algebra $\mathfrak{g}$ and half-hypermultiplet representation content $\mathbf{r}$ by $\mathfrak{g}$ w/ $\mathbf{r}$, from which the Coulomb branch vev dimensions and the flavor algebra can be determined.[1] In this notation, the duality proposed in [1] has the general form

$$\mathfrak{g}[d_i] \text{ w/ } \mathbf{r} = \widetilde{\mathfrak{g}}[\widetilde{d}_i] \text{ w/ } (\widetilde{\mathbf{r}} \oplus \text{SCFT}[d : \mathfrak{h}]), \tag{1}$$

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[1]The rules: The Coulomb branch vevs are the orders of the independent adjoint Casimirs of $\mathfrak{g}$, which are the exponents of $\mathfrak{g}$ plus 1. The flavor symmetry depends on whether the half-hypermultiplets are in complex, real (orthogonal), or pseudoreal (symplectic) representations. A complex representation always appears with its complex conjugate, and $n \cdot (\mathbf{r} \oplus \overline{\mathbf{r}})$ contributes a u(n) flavor symmetry factor. Only even numbers of half-hypermultiplets in real representations can be coupled, and $2n \cdot \mathbf{r}$ contributes an sp(n) flavor symmetry factor. Any number of pseudoreal half-hypermultiplets can be coupled, and $n \cdot \mathbf{r}$ gives an so(n) flavor symmetry factor.

d	$\mathfrak{h}$	$k_{\mathfrak{h}}$	$\frac{3}{2} \cdot k_R$	$48 \cdot a$	$\mathbb{Z}_2$ obstruction?
6	E_8	12	124	190	no
6	$\mathrm{sp}(5)$	7	98	164	yes
4	E_7	8	76	118	no
4	$\mathrm{sp}(3) \oplus \mathrm{su}(2)$	$5 \oplus 8$	58	100	yes $\oplus$ no
3	E_6	6	52	82	no
3	$2 \leq \mathrm{rank}(\mathfrak{h}) \leq 6$	≤ 8	34–38	64–68	?

Table 1: Properties of predicted rank 1 SCFTs. d is the dimension of the Coulomb branch vev, $\mathfrak{h}$ is the flavor symmetry algebra, $k_{\mathfrak{h}}$ is the flavor current algebra central charge, k_R is the $\mathrm{u}(1)_R$ current algebra central charge, and a is one of the conformal anomalies. Only ranges of possible values are given for the entry in the last row.

where we have indicated the dimensions of the Coulomb branch vevs of the Lagrangian gauge groups as well, for clarity.

Our main results are the existence and properties of the rank one isolated SCFTs shown in table 1. The curves of the $E_{6,7,8}$ theories were found in [3]. The central charges of the E_6 and E_7 theories were found by consistency of the infinite-coupling duality proposal in [1]. In the rest of this note we use similar consistency arguments to compute the other entries in table 1. These properties agree with those of the $E_{6,7,8}$ theories computed using AdS/CFT techniques in [6].

The evidence for the new SCFTs comes from finding many different examples of dualities of the form (1) with SCFT factors with the properties shown in table 1. As we describe in more detail below, we search through a list of Lagrangian conformal theories with the assumption that they have infinite coupling duals of the form (1), and compute the properties of the assumed isolated SCFT by matching various symmetries and anomalies on the two sides of the duality. Whenever we find two or more examples giving matching SCFTs, we include it in table 1. In fact, the resulting set of dualities gives many examples for each of the theories, consistent in an intricate and beautiful way, and are listed in tables 2 and 3 below. The exception is the SCFT in the last line of table 1 for which there is only one duality which does not give enough constraints to pin down its properties precisely. In an appendix we include some notes on computing Lie algebra embeddings, and record the embeddings used in tables 2 and 3, for the convenience of readers interested in checking our results.

Constraints on possible infinite coupling duals of the form (1) come from matching on both sides of the equivalence the following seven quantities:

   i The rank of the gauge group and the spectrum of dimensions of the Coulomb branch vevs, implying

$$\{d_i\} = \{\widetilde{d_i}\} \cup \{d\}. \tag{2}$$

   ii The flavor symmetry algebras, implying the flavor symmetry on the left is the sum, $\widetilde{\mathfrak{f}} \oplus \mathfrak{f}$, of the flavor symmetry $\widetilde{\mathfrak{f}}$ of the $\widetilde{\mathbf{r}}$ half-hypermultiplets on the right and the commutant

of $\widetilde{\mathfrak{g}}$ in $\mathfrak{h}$,

$$\widetilde{\mathfrak{g}} \oplus \mathfrak{f} \subset \mathfrak{h} \quad \text{with } \mathfrak{f} \text{ maximal.} \tag{3}$$

iii The contribution to the beta function from weakly gauging the flavor symmetry on both sides, giving

$$T(\mathbf{r}) = T(\widetilde{\mathbf{r}}) + k_{\mathfrak{h}} \cdot I_{\mathfrak{f}\to\mathfrak{h}}, \tag{4}$$

where $T(\mathbf{r})$ is the quadratic index of the representation $\mathbf{r}$, $I_{\mathfrak{f}\to\mathfrak{h}}$ is the Dynkin index of embedding, and $k_{\mathfrak{h}}$ is the central charge of the $\mathfrak{h}$ flavor symmetry current algebra.

iv The number of marginal couplings, implying the vanishing of the beta function of the $\widetilde{\mathfrak{g}}$ gauge factor,

$$2 \cdot T(\widetilde{\mathfrak{g}}) = T(\widetilde{\mathbf{r}}) + k_{\mathfrak{h}} \cdot I_{\widetilde{\mathfrak{g}}\to\mathfrak{h}}, \tag{5}$$

where $T(\widetilde{\mathfrak{g}})$ denotes the quadratic index of the adjoint representation of $\widetilde{\mathfrak{g}}$.

v The contribution to the $\mathfrak{u}(1)_R$ symmetry central charge on both sides, giving this central charge of the SCFT, k_R, as

$$(3/2) \cdot k_R = 24 \cdot c = 4 \cdot (|\mathfrak{g}| - |\widetilde{\mathfrak{g}}|) + (|\mathbf{r}| - |\widetilde{\mathbf{r}}|), \tag{6}$$

which is related as shown to the energy-momentum central charge c. $|\mathfrak{g}|$ and $|\mathbf{r}|$ are the dimensions of the algebra and representation, respectively.

vi The contribution to the a conformal anomaly, giving a for the SCFT as

$$48 \cdot a = 10 \cdot (|\mathfrak{g}| - |\widetilde{\mathfrak{g}}|) + (|\mathbf{r}| - |\widetilde{\mathbf{r}}|). \tag{7}$$

In general $|\mathfrak{g}| = \sum_i (2d_i - 1)$, implying from (2) that $|\mathfrak{g}| - |\widetilde{\mathfrak{g}}| = 2d - 1$. So the difference between (6) and (7) is already fixed by condition i.

vii Whether there is a global $\mathbb{Z}_2$ obstruction [7] to gauging of the flavor symmetry.

The first five conditions were described in [1], while the a and c conformal anomalies can be computed from 't Hooft anomaly matching, as reviewed in [6]. The global $\mathbb{Z}_2$ obstruction matching is described below. Our conventions for the normalization of the central charges and for the quadratic index follow those of [1].

Let us illustrate the use of these constraints in determining the dual SCFT in the case of the the Lagrangian G_2 w/ $8 \cdot \mathbf{7}$ conformal theory, given as entry 4 in table 2. We assume it has an infinite-coupling dual of the form (1). Apply the above list of conditions:

i Since G_2 is rank 2, with adjoint Casimirs of order 2 and 6, the only possibility is that SCFT[$\mathfrak{h}$] and $\widetilde{\mathfrak{g}}$ are both rank 1 gauge theories. Thus we must have $\widetilde{\mathfrak{g}} = \mathfrak{su}(2)$, and since

its Coulomb branch vev has dimension 2, the dimension 6 vev must belong to the rank 1 SCFT$[6 : \mathfrak{h}]$.[2] Thus

$$G_2 \text{ w/ } 8 \cdot \mathbf{7} = \mathrm{su}(2) \text{ w/ } (n \cdot \mathbf{2} \oplus \mathrm{SCFT}[6 : \mathfrak{h}]) \tag{8}$$

with $\mathrm{su}(2) \subset \mathfrak{h}$. Only $n \leq 7$ half-hypermultiplet $\mathbf{2}$'s can occur since any more or any other representations would contribute too much to the $\mathrm{su}(2)$ beta function, making it infrared free.

ii The left side of (8) has $\mathrm{sp}(4)$ flavor symmetry since the $\mathbf{7}$ is a real representation. Since the $\mathbf{2}$ of $\mathrm{su}(2)$ is pseudoreal, the half-hypermultiplets on the right side contribute an $\mathrm{so}(n)$ flavor symmetry factor. Thus the only way to match the global symmetries on both sides is for $n = 0$ or 1 (so that the half-hypermultiplets don't contribute any flavor factors), and $\mathrm{su}(2) \oplus \mathrm{sp}(4) \subset \mathfrak{h}$ with $\mathrm{sp}(4)$ being the commutant of $\mathrm{su}(2)$ in $\mathfrak{h}$. Assuming that $\mathfrak{h}$ is simple and has rank less than or equal to 8, some work with a table of maximal subalgebras [10] shows that only $\mathrm{so}(16)$ and $\mathrm{sp}(5)$ have $\mathrm{su}(2)$ subalgebras with commutant $\mathrm{sp}(4)$. In either case the $\mathrm{sp}(4)$ factor has Dynkin index of embedding $I_{\mathrm{sp}(4) \hookrightarrow \mathrm{so}(16)} = I_{\mathrm{sp}(4) \hookrightarrow \mathrm{sp}(5)} = 1$, but $I_{\mathrm{su}(2) \hookrightarrow \mathrm{so}(16)} = 4$, while $I_{\mathrm{su}(2) \hookrightarrow \mathrm{sp}(5)} = 1$.

iii For the $\mathrm{sp}(4)$ flavor symmetry $7 = 7 \cdot T(\mathbf{8}) = k_{\mathfrak{h}} I_{\mathrm{sp}(4) \hookrightarrow \mathfrak{h}}$. (The $n \cdot \mathbf{2}$'s don't contribute because they are singlets under the $\mathrm{sp}(4)$.) Since the index of embedding is 1, we find that $k_{\mathfrak{h}} = 7$ whether $\mathfrak{h} = \mathrm{so}(16)$ or $\mathrm{sp}(5)$.

iv $8 = T(\mathrm{su}(2)) = n + 7 \cdot I_{\mathrm{su}(2) \hookrightarrow \mathfrak{h}}$. Since $n \in \{0, 1\}$ and the index of embedding is a positive integer, the only solution is $n = 1$ and $I_{\mathrm{su}(2) \hookrightarrow \mathfrak{h}} = 1$. Thus we must have $\mathfrak{h} = \mathrm{sp}(5)$.

v $(3/2) \cdot k_R = 4 \cdot (14 - 3) + (7 \cdot 8 - 1 \cdot 2) = 98$.

vi $48 \cdot a = 10 \cdot (14 - 3) + (7 \cdot 8 - 1 \cdot 2) = 164$.

vii The $\mathrm{sp}(4)$ flavor symmetry of G_2 w/ $8 \cdot \mathbf{7}$ has a global $\mathbb{Z}_2$ obstruction to being gauged since there is an odd number (7) of the pseudoreal $\mathbf{8}$'s of $\mathrm{sp}(4)$. So, to weakly gauge the $\mathrm{sp}(4)$, a spectator half-hypermultiplet (or just a Weyl fermion) in a pseudoreal representation must be added. Since the spectator fields are otherwise uncoupled, this $\mathbb{Z}_2$ obstruction must persist whatever the value of the G_2 coupling, and so should also be seen in the dual $\mathrm{su}(2)$ w/ $(\mathbf{2} \oplus \mathrm{SCFT}[6 : \mathrm{sp}(5)])$ theory. Since the SCFT factor contributes the fields transforming under $\mathrm{sp}(4)$, and since $I_{\mathrm{sp}(4) \hookrightarrow \mathrm{sp}(5)} = 1$ is odd, it follows that the $\mathrm{SCFT}^{[6]}_{\mathrm{sp}(5)}$ theory by itself must have a global $\mathbb{Z}_2$ obstruction in its

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[2]In this particular case, though not in most of the other cases we consider, there is strong additional evidence for these conclusions: the form of the curve describing the low energy effective action of the G_2 theory is known [8], from which it is easily checked that it factorizes into the scale invariant $\mathrm{su}(2)$ singularity [9] and the SCFT[6] singularity in the infinite coupling limit.

sp(5) flavor algebra.[3] This is necessary also for the su(2) gauge factor to be anomaly-free: the single **2** half-hypermultiplet contributes a $\mathbb{Z}_2$ anomaly, but so does the SCFT factor since su(2) is of odd index in sp(5).

We assumed in the above argument that rank($\mathfrak{h}$) ≤ 8 and $\mathfrak{h}$ was simple. The first assumption is justified because the maximum number of independent mass deformations of SCFT[6] is 8—as seen by counting the independent deformations of the complex structure of its curve—so whatever its flavor algebra it must have rank at most 8. The simplicity of $\mathfrak{h}$ assumption has no justification, and without it there are many more possible answers. For example, we could have $\mathfrak{h} = $ su(2) $\oplus$ sp(4). In these non-simple cases, each simple factor can have a different central charge, making it easy to satisfy the requirements to be a dual description.

Even though each individual assumed duality may be consistent with more than one possible isolated rank 1 SCFT, we can gain stronger evidence for the existence of a particular one by showing that it consistently occurs in the duals of other theories. We can check for this by examining higher-rank Lagrangian conformal theories which could have infinite coupling duals with SCFT[d] factors with $d = 3$, 4, or 6. The possible Lagrangian theories are constrained by requirement (2) above. For example, for the infinite-coupling dual to contain a SCFT[6] factor, the gauge algebra $\mathfrak{g}$ of the Lagrangian theory must include an order 6 adjoint Casimir, and its list of remaining Casimir orders must be those of the dual gauge algebra $\widetilde{\mathfrak{g}}$. There are only a handful of possibilities for such $\mathfrak{g}$ among simple algebras, namely su(3), sp(2), G_2, su(4), so(7), sp(3), so(8), su(6), and so(12). From the known curves for the rank 2 theories, only 6 have infinite-coupling limits [1], and there are 53 (non-N=4) Lagrangian conformal theories with gauge algebras, $\mathfrak{g}$, with rank ≥ 3 in this list. By searching through this list for pairs of theories consistent with duals involving the same SCFTs, we find the 16 ones shown in table 2, all consistent with the properties recorded in table 1. Some details of the flavor symmetry embeddings for each entry in table 2 are collected in the appendix.

That not all of the 53 rank ≥ 3 Lagrangian theories appear in table 2 is reasonable, since they need not all necessarily have rank 1 SCFT factors: they could be self-dual, or could be dual to theories with rank ≥ 2 SCFT factors. For example, from the known curves for the Lagrangian superconformal theories with rank r classical gauge groups with hypermultiplets in fundamental representations [11], it follows from their degeneration in the infinite coupling limit that their duals are su(2) plus rank $r - 1$ isolated SCFTs.

Entry 17 of table 2 is the only case where we do not have enough information to determine the properties of the dual SCFT precisely. Because the flavor symmetry is u(1) $\oplus$ u(1), we can only have $n \in \{0, 1, 2\}$, and must have su(2)$_I \oplus$ u(1) $\subset \mathfrak{h}$, with u(1) maximal and I the index of embedding of the su(2). Then $k_{\mathfrak{h}} = (8 - n)/I$, $(3/2) \cdot k_R = 38 - 2n$, and $48 \cdot a = 68 - 2n$. There are many possibilities for $\mathfrak{h}$ consistent with the embedding constraint. Nevertheless, since k_R and a are smaller than those of the SCFT[3 : E_6] theory, and since it is known from its curve [12] that the su(3) w/ $\mathbf{3} \oplus \overline{\mathbf{3}} \oplus \mathbf{6} \oplus \overline{\mathbf{6}}$ theory has an infinite coupling limit, it follows that there must be a new SCFT[3].

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[3]Thanks to N. Seiberg for discussions on this point.

	$\mathfrak{g}$ w/	$\mathbf{r}$	= $\widetilde{\mathfrak{g}}$ w/	$\widetilde{\mathbf{r}}$	$\oplus$	$\mathrm{SCFT}[d:\mathfrak{h}]$
1.	$\mathrm{sp}(3)$	$\mathbf{14}\oplus 11\cdot\mathbf{6}$	= $\mathrm{sp}(2)$			$[6:E_8]$
2.	$\mathrm{su}(6)$	$\mathbf{20}\oplus\mathbf{15}\oplus\overline{\mathbf{15}}\oplus 5\cdot\mathbf{6}\oplus 5\cdot\overline{\mathbf{6}}$	= $\mathrm{su}(5)$	$\mathbf{5}\oplus\overline{\mathbf{5}}\oplus\mathbf{10}\oplus\overline{\mathbf{10}}$		$[6:E_8]$
3.	$\mathrm{so}(12)$	$3\cdot\mathbf{32}\oplus\mathbf{32'}\oplus 4\cdot\mathbf{12}$	= $\mathrm{so}(11)$	$3\cdot\mathbf{32}$		$[6:E_8]$
4.	G_2	$8\cdot\mathbf{7}$	= $\mathrm{su}(2)$	$\mathbf{2}$		$[6:\mathrm{sp}(5)]$
5.	$\mathrm{so}(7)$	$4\cdot\mathbf{8}\oplus 6\cdot\mathbf{7}$	= $\mathrm{sp}(2)$	$5\cdot\mathbf{4}$		$[6:\mathrm{sp}(5)]$
6.	$\mathrm{su}(6)$	$\mathbf{21}\oplus\overline{\mathbf{21}}\oplus\mathbf{20}\oplus\mathbf{6}\oplus\overline{\mathbf{6}}$	= $\mathrm{su}(5)$	$\mathbf{10}\oplus\overline{\mathbf{10}}$		$[6:\mathrm{sp}(5)]$
7.	$\mathrm{sp}(2)$	$12\cdot\mathbf{4}$	= $\mathrm{su}(2)$			$[4:E_7]$
8.	$\mathrm{su}(4)$	$2\cdot\mathbf{6}\oplus 6\cdot\mathbf{4}\oplus 6\cdot\overline{\mathbf{4}}$	= $\mathrm{su}(3)$	$2\cdot\mathbf{3}\oplus 2\cdot\overline{\mathbf{3}}$		$[4:E_7]$
9.	$\mathrm{so}(7)$	$6\cdot\mathbf{8}\oplus 4\cdot\mathbf{7}$	= G_2	$4\cdot\mathbf{7}$		$[4:E_7]$
10.	$\mathrm{so}(8)$	$6\cdot\mathbf{8}\oplus 4\cdot\mathbf{8'}\oplus 2\cdot\mathbf{8''}$	= $\mathrm{so}(7)$	$6\cdot\mathbf{8}$		$[4:E_7]$
11.	$\mathrm{so}(8)$	$6\cdot\mathbf{8}\oplus 6\cdot\mathbf{8'}$	= G_2			$[4:E_7]\oplus[4:E_7]$
12.	$\mathrm{sp}(2)$	$6\cdot\mathbf{5}$	= $\mathrm{su}(2)$			$[4:\mathrm{sp}(3)\oplus\mathrm{su}(2)]$
13.	$\mathrm{sp}(2)$	$4\cdot\mathbf{4}\oplus 4\cdot\mathbf{5}$	= $\mathrm{su}(2)$	$3\cdot\mathbf{2}$		$[4:\mathrm{sp}(3)\oplus\mathrm{su}(2)]$
14.	$\mathrm{su}(4)$	$\mathbf{10}\oplus\overline{\mathbf{10}}\oplus 2\cdot\mathbf{4}\oplus 2\cdot\overline{\mathbf{4}}$	= $\mathrm{su}(3)$	$\mathbf{3}\oplus\overline{\mathbf{3}}$		$[4:\mathrm{sp}(3)\oplus\mathrm{su}(2)]$
15.	$\mathrm{su}(3)$	$6\cdot\mathbf{3}\oplus 6\cdot\overline{\mathbf{3}}$	= $\mathrm{su}(2)$	$2\cdot\mathbf{2}$		$[3:E_6]$
16.	$\mathrm{su}(4)$	$4\cdot\mathbf{6}\oplus 4\cdot\mathbf{4}\oplus 4\cdot\overline{\mathbf{4}}$	= $\mathrm{sp}(2)$	$6\cdot\mathbf{4}$		$[3:E_6]$
17.	$\mathrm{su}(3)$	$\mathbf{3}\oplus\overline{\mathbf{3}}\oplus\mathbf{6}\oplus\overline{\mathbf{6}}$	= $\mathrm{su}(2)$	$n\cdot\mathbf{2}$		$[3:\mathfrak{h}]$

Table 2: Predicted dualities with one marginal operator.

There is another, less direct, way of constraining $\mathfrak{h}$ for entry 17. We can reverse the direction of our logic, and, starting with SCFT$[3:\mathfrak{h}]$ we can gauge different su(2) subalgebras of $\mathfrak{h}$ and try to add appropriate numbers of doublet half-hypermultiplets to make the gauge coupling marginal. By construction there will be one such embedding giving the Lagrangian conformal theory of entry 17. But if there are other embeddings, then we would predict the existence of a rank 2 conformal theory with Coulomb branch dimensions 2 and 3 and with a marginal coupling, but no purely weakly coupled (Lagrangian) limit. There is nothing wrong with this in general—indeed, their existence for higher ranks is a robust prediction of infinite-coupling duality, as we will discuss below. However, at rank 2 there is some evidence from systematic searches for all possible curves of rank 2 SCFTs [8, 13] that all those curves with marginal operators have a limit in the coupling space where the curve becomes singular in a way consistent with a purely weakly coupled Lagrangian description. This is further supported by the fact that for the other 5 rank 1 SCFTs listed in table 1, the only conformal gaugings of su(2) subalgebras are precisely those with Lagrangian limits. If we assume then that this should also apply to the SCFT$[3:\mathfrak{h}]$ theory, it then follows from a detailed examination of algebra embeddings that $\mathrm{rank}(\mathfrak{h})=2$, since all higher-rank $\mathfrak{h}$ turn out to admit multiple inequivalent conformal gaugings of su(2) subalgebras. Assuming $\mathrm{rank}(\mathfrak{h})=2$ forces $n=2$, so gives $k_\mathfrak{h}=6/I$, $(3/2)\cdot k_R=34$ and $48\cdot a=64$, but does not constrain $\mathfrak{h}$ any further since all rank 2 $\mathfrak{h}$'s only admit the single Lagrangian conformal su(2) embedding.

	$\mathfrak{g}$	w/ $\mathbf{r}$	$=$	$\widetilde{\mathfrak{g}}$	w/ $\widetilde{\mathbf{r}}$	$\oplus$	SCFT$[d:\mathfrak{h}]$
18.	$su(2)\oplus su(3)$	$2\cdot(\mathbf{2},\mathbf{1})\oplus(\mathbf{2},\mathbf{3}\oplus\overline{\mathbf{3}})\oplus4\cdot(\mathbf{1},\mathbf{3}\oplus\overline{\mathbf{3}})$	$=$	$su(2)\oplus su(2)$	$2\cdot(\mathbf{2},\mathbf{1})\oplus2\cdot(\mathbf{1},\mathbf{2})$		$[3:E_6]$
19.	$su(2)\oplus sp(2)$	$2\cdot(\mathbf{2},\mathbf{4})\oplus8\cdot(\mathbf{1},\mathbf{4})$	$=$	$su(2)\oplus su(2)$			$[4:E_7]$
20.	$su(2)\oplus sp(2)$	$3\cdot(\mathbf{2},\mathbf{1})\oplus(\mathbf{2},\mathbf{5})\oplus4\cdot(\mathbf{1},\mathbf{5})$	$=$	$su(2)\oplus su(2)$	$3\cdot(\mathbf{2},\mathbf{1})$		$[4:sp(3)\oplus su(2)]$
21.	$su(2)\oplus G_2$	$(\mathbf{2},\mathbf{1})\oplus(\mathbf{2},\mathbf{7})\oplus6\cdot(\mathbf{1},\mathbf{7})$	$=$	$su(2)\oplus su(2)$	$(\mathbf{2},\mathbf{1})\oplus(\mathbf{1},\mathbf{2})$		$[6:sp(5)]$
22.	$su(3)\oplus su(3)$	$2\cdot(\mathbf{3},\overline{\mathbf{3}})\oplus2\cdot(\overline{\mathbf{3}},\mathbf{3})$	$=$	$su(2)\oplus su(3)$	$2\cdot(\mathbf{2},\mathbf{1})$		$[3:E_6]$
23.	$su(3)\oplus su(3)$	$(\mathbf{3}\oplus\overline{\mathbf{3}},\mathbf{3}\oplus\overline{\mathbf{3}})$	$=$	$su(2)\oplus su(3)$	$2\cdot(\mathbf{2},\mathbf{1})$		$[3:E_6]$
24.	$su(3)\oplus su(3)$	$3\cdot(\mathbf{3}\oplus\overline{\mathbf{3}},\mathbf{1})\oplus(\mathbf{3},\overline{\mathbf{3}})\oplus(\overline{\mathbf{3}},\mathbf{3})$	$=$	$su(2)\oplus su(3)$	$2\cdot(\mathbf{2},\mathbf{1})$		$[3:E_6]$
		$\oplus3\cdot(\mathbf{1},\mathbf{3}\oplus\overline{\mathbf{3}})$			$\oplus3\cdot(\mathbf{1},\mathbf{3}\oplus\overline{\mathbf{3}})$		
25.	$su(3)\oplus sp(2)$	$(\mathbf{3}\oplus\overline{\mathbf{3}},\mathbf{1})\oplus(\mathbf{3}\oplus\overline{\mathbf{3}},\mathbf{5})$	$=$	$su(2)\oplus sp(2)$	$2\cdot(\mathbf{2},\mathbf{1})$		$[3:E_6]$
			$=$	$su(3)\oplus su(2)$	$(\mathbf{3}\oplus\overline{\mathbf{3}},\mathbf{1})$		$[4:sp(3)\oplus su(2)]$
26.	$su(3)\oplus sp(2)$	$2\cdot(\mathbf{3}\oplus\overline{\mathbf{3}},\mathbf{1})\oplus(\mathbf{3}\oplus\overline{\mathbf{3}},\mathbf{4})\oplus6\cdot(\mathbf{1},\mathbf{4})$	$=$	$su(2)\oplus sp(2)$	$2\cdot(\mathbf{2},\mathbf{1})\oplus6\cdot(\mathbf{1},\mathbf{4})$		$[3:E_6]$
			$=$	$su(3)\oplus su(2)$	$2\cdot(\mathbf{3}\oplus\overline{\mathbf{3}},\mathbf{1})$		$[4:E_7]$
27.	$sp(2)\oplus sp(2)$	$2\cdot(\mathbf{5},\mathbf{1})\oplus(\mathbf{5},\mathbf{4})\oplus7\cdot(\mathbf{1},\mathbf{4})$	$=$	$su(2)\oplus sp(2)$	$7\cdot(\mathbf{1},\mathbf{4})$		$[4:sp(3)\oplus su(2)]$
			$=$	$sp(2)\oplus su(2)$	$2\cdot(\mathbf{5},\mathbf{1})$		$[4:E_7]$
28.	$sp(2)\oplus sp(2)$	$4\cdot(\mathbf{4},\mathbf{1})\oplus2\cdot(\mathbf{4},\mathbf{4})\oplus4\cdot(\mathbf{1},\mathbf{4})$	$=$	$su(2)\oplus sp(2)$	$4\cdot(\mathbf{1},\mathbf{4})$		$[4:E_7]$
29.	$sp(2)\oplus G_2$	$5\cdot(\mathbf{4},\mathbf{1})\oplus(\mathbf{4},\mathbf{7})\oplus4\cdot(\mathbf{1},\mathbf{7})$	$=$	$su(2)\oplus G_2$	$4\cdot(\mathbf{1},\mathbf{7})$		$[4:E_7]$
			$=$	$sp(2)\oplus su(2)$	$5\cdot(\mathbf{4},\mathbf{1})\oplus(\mathbf{1},\mathbf{2})$		$[6:sp(5)]$

Table 3: Some predicted dualities with two marginal operators.

In general there are many different ways of gauging rank ≥ 2 subalgebras of the flavor algebras of the SCFTs in table 1. This provides many examples of conformal theories with marginal couplings which do not appear in the list in table 2 of such theories with purely Lagrangian descriptions in some limit. Some simple examples are

$$
\begin{aligned}
su(3) \text{ w/ SCFT}[3:E_6] && E_6 \supset su(3)_2 \oplus (G_2)_1, \\
su(3) \text{ w/ SCFT}[6:E_8] && E_8 \supset su(3)_1 \oplus (E_6)_1, \\
G_2 \text{ w/ } 2\cdot\mathbf{7} \oplus \text{SCFT}[6:E_8] && E_8 \supset (G_2)_1 \oplus (F_4)_1, \\
F_4 \text{ w/ } 4\cdot\mathbf{26} \oplus \text{SCFT}[6:E_8] && E_8 \supset (F_4)_1 \oplus (G_2)_1,
\end{aligned}
\tag{9}
$$

where we have indicated the embedding of the gauge algebra in the SCFT flavor algebra on the right, with the index of embedding of each subalgebra shown as a subscript. For these examples it is clear that they cannot be dual to a purely Lagrangian field theory, since they all have exceptional flavor algebras. These are thus examples of conformal theories with marginal couplings but no purely weak-coupling limit.

All our arguments can also be applied to theories with more than one marginal gauge coupling. We illustrate this on the small set of rank 2 and 3 Lagrangian theories with two marginal couplings which reduce in the limit as one of the couplings becomes weak to one of the rank 2 theories (entries 4, 7, 12, 13, 15, or 17) in table 2. In the rank 3 cases, one factor of the gauge algebra must be $su(2)$ which is self dual. The dual descriptions in the infinite-coupling limit of the other factor's coupling are given in entries 18 to 21 of table 3. For the rank 4 examples, where each simple gauge algebra factor has rank 2, we give their dual descriptions in table 3 only in the limit as one or the other gauge factor is taken to

infinite coupling. (If the two infinite coupling limits are different, we give the dual of the first factor on the first line and that of the second factor on the second line of the entry in table 3.) The double infinite-coupling limit of all these theories have dual descriptions of the form $\mathrm{su}(2) \oplus \mathrm{su}(2)$ w/ $\mathbf{r} \oplus \mathrm{SCFT}[d_1, d_2]$ where $\mathrm{SCFT}[d_1, d_2]$ are isolated rank 2 SCFTs. The success of the infinite coupling dual descriptions of all these theories is strong additional evidence for the existence of the first five SCFTs in table 1.

The existence of the new rank 1 SCFTs with flavor symmetries other than E_n raises some obvious questions: What are the Seiberg-Witten curves and one-forms describing the low energy effective actions of (the mass deformations of) these theories? Are there string constructions that realize these sub-maximal SCFTs?

The set of dual descriptions found in tables 2 and 3, together with the known Seiberg-Witten singular curves for the rank 1 SCFTs and some Lagrangian conformal theories that appear on the right side of the duality can be used in many cases to determine the singular curves for the higher-rank Lagrangian theories on the left.

The techniques of this paper can also be applied to determining the properties of higher-rank isolated SCFTs. Here, though, the story will certainly be much more complicated: there are many known rank 2 singularities [8, 13], a complete list has not been found, and techniques for determining their mass deformations consistent with the requirements of $N = 2$ supersymmetry have not been developed.

Acknowledgments

Appendix

The embeddings $\widetilde{\mathfrak{g}} \subset \mathfrak{h}$ for each of the dual theories listed in tables 2 and 3 are listed below in table 4, where the Dynkin index of embedding of each subalgebra is shown as a subscript. We make a few comments on how these embeddings can be extracted from tables. Tables of maximal semisimple subalgebras of simple Lie algebras are given in [10]. We need not only the semisimple factors, but also any u(1) factors that may occur. Maximal reductive subalgebras of simple Lie algebras which have an abelian factor only have a single u(1), and their semisimple factors correspond to the Dynkin diagram which results from eliminating any two nodes with mark 1 from the extended Dynkin diagram of the original algebra [14].

All embeddings can be found by following chains of maximal embeddings. One is then faced with finding maximal embeddings in semi-simple algebras. The only non-trivial case then are diagonal embeddings in two or more identical factors. For example, $\mathfrak{h} \supset \mathfrak{g}_J \oplus \mathfrak{g}_K \supset$

$\mathfrak{g}_{J+K} \oplus \mathrm{u}(1) \oplus \cdots \oplus \mathrm{u}(1)$ where there are $\mathrm{rank}(\mathfrak{g})$ $\mathrm{u}(1)$ factors; the subscripts are to indicate that the Dynkin indices of embedding add under diagonal embedding.

The problem we face is not just to find an embedding of a gauge algebra $\widetilde{\mathfrak{g}}$ in $\mathfrak{h}$, but also to compute the commutant of $\widetilde{\mathfrak{g}}$ in $\mathfrak{h}$. In general, upon following a chain of maximal embeddings, the resulting subalgebra commuting with $\widetilde{\mathfrak{g}}$ need not be maximal. However, the maximal commuting subalgebra will appear somewhere in the tree of all possible chains of maximal embeddings which contain $\widetilde{\mathfrak{g}}$ as a factor. To determine whether a given commuting subalgebra is maximal or is itself a subalgebra of a commuting subalgebra found in a different chain, one needs to check whether the two different embeddings of $\widetilde{\mathfrak{g}}$ are equivalent or not.

For example, consider the case of entries 22 and 23 in table 3. These are two different Lagrangian theories whose duals have the same gauge group $\widetilde{\mathfrak{g}} = \mathrm{su}(2)_1 \oplus \mathrm{su}(3)_2$ both embedded in the flavor algebra $\mathfrak{h} = E_6$ of SCFT$[3 : E_6]$. (Theory 22 is an "elliptic model" whose low energy effective action was computed in [15], while theory 23 is a "twisted elliptic" model whose curve is not known.) In the case of theory 22 the chain of maximal embeddings is

$$E_6 \supset \mathrm{su}(3)_2 \oplus \left((G_2)_1 \supset \mathrm{su}(2)_1 \oplus \mathrm{su}(2)_3 \right), \tag{10}$$

while for theory 23 it is

$$E_6 \supset \mathrm{su}(2)_1 \oplus \left(\mathrm{su}(5)_1 \supset \mathrm{su}(4)_2 \supset \mathrm{su}(3)_2 \oplus \mathrm{u}(1) \right). \tag{11}$$

To show that these are really different maximal embeddings, we must show that they are different embeddings of $\mathrm{su}(2)_1 \oplus \mathrm{su}(3)_2$ in E_6, otherwise (11) would just be a subalgebra of (10). To show this, we check the branching of a specific representation. For (10), under $E_6 \supset \mathrm{su}(3) \oplus G_2$: $\mathbf{27} = (\overline{\mathbf{6}}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{7})$, and under $G_2 \supset \mathrm{su}(2)_1 \oplus \mathrm{su}(2)_3$: $\mathbf{7} = (\mathbf{2}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{3})$. For (11), under $E_6 \supset \mathrm{su}(2)_1 \oplus \mathrm{su}(5)_1$: $\mathbf{27} = (\mathbf{1}, \overline{\mathbf{15}}) \oplus (\mathbf{2}, \mathbf{6})$, and under $\mathrm{su}(6) \supset \mathrm{su}(3) \supset \mathrm{su}(2) \oplus \mathrm{u}(1)$: $\overline{\mathbf{15}} = \mathbf{15} = \mathbf{8} \oplus \mathbf{3} \oplus \overline{\mathbf{3}} \oplus \mathbf{1}$ and $\mathbf{6} = \mathbf{6} = \mathbf{3} \oplus \overline{\mathbf{3}}$. Putting these together, under the (10) embedding

$$E_6 \supset \mathrm{su}(3)_2 \oplus \mathrm{su}(2)_1 \quad : \quad \mathbf{27} = (\overline{\mathbf{6}}, \mathbf{1}) \oplus 2 \cdot (\mathbf{3}, \mathbf{2}) \oplus 3 \cdot (\mathbf{3}, \mathbf{1}),$$

while under the (11) embedding

$$E_6 \supset \mathrm{su}(3)_2 \oplus \mathrm{su}(2)_1 \quad : \quad \mathbf{27} = (\mathbf{8}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{1}) \oplus (\overline{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{2}) \oplus (\overline{\mathbf{3}}, \mathbf{2}),$$

showing that they are inequivalent embeddings.

References

[1] P. C. Argyres and N. Seiberg, "S-duality in N=2 supersymmetric gauge theories," arXiv:0711.0054 [hep-th].

[2] P. C. Argyres, M. R. Plesser, N. Seiberg and E. Witten, *Nucl. Phys.* **B 461** (1996) 71 [`hep-th/9511154`]

	$\mathfrak{h}$	$\supset \widetilde{\mathfrak{g}}_I$	$\oplus \mathfrak{f}_{I'}$
1.	E_8	$\supset \mathrm{sp}(2)_1$	$\oplus \mathrm{so}(11)_1$
2.	E_8	$\supset \mathrm{su}(5)_1$	$\oplus \mathrm{su}(5)_1$
3.	E_8	$\supset \mathrm{so}(11)_1$	$\oplus \mathrm{sp}(2)_1$
4.	$\mathrm{sp}(5)$	$\supset \mathrm{su}(2)_1$	$\oplus \mathrm{sp}(4)_1$
5.	$\mathrm{sp}(5)$	$\supset \mathrm{sp}(2)_1$	$\oplus \mathrm{sp}(3)_1$
6.	$\mathrm{sp}(5)$	$\supset \mathrm{su}(5)_2$	$\oplus \mathrm{u}(1)$
7.	E_7	$\supset \mathrm{su}(2)_1$	$\oplus \mathrm{so}(12)_1$
8.	E_7	$\supset \mathrm{su}(3)_1$	$\oplus \mathrm{su}(6)_1$
9.	E_7	$\supset (G_2)_1$	$\oplus \mathrm{sp}(3)_1$
10.	E_7	$\supset \mathrm{so}(7)_1$	$\oplus \mathrm{sp}(2)_1 \oplus \mathrm{su}(2)_1$
11.	E_7	$\supset (G_2)_1$	$\oplus \mathrm{sp}(3)_1 \qquad$ for both
12.	$\mathrm{sp}(3) \oplus \underline{\mathrm{su}(2)}$	$\supset \underline{\mathrm{su}(2)_1}$	$\oplus \mathrm{sp}(3)_1$
13.	$\mathrm{sp}(3) \oplus \underline{\mathrm{su}(2)}$	$\supset \underline{\mathrm{su}(2)_1}$	$\oplus \mathrm{sp}(2)_1 \oplus \underline{\mathrm{su}(2)_1}$
14.	$\mathrm{sp}(3) \oplus \underline{\mathrm{su}(2)}$	$\supset \mathrm{su}(3)_2$	$\oplus \mathrm{u}(1) \oplus \underline{\mathrm{su}(2)_1}$
15.	E_6	$\supset \mathrm{su}(2)_1$	$\oplus \mathrm{su}(6)_1$
16.	E_6	$\supset \mathrm{sp}(2)_1$	$\oplus \mathrm{sp}(2)_1 \oplus \mathrm{u}(1)$
17.	$\mathfrak{h}$	$\supset \mathrm{su}(2)_I$	$\oplus \mathrm{u}(1) \qquad$ if $n = 2$
		$\supset \mathrm{su}(2)_I$	$\oplus \mathrm{u}(1) \oplus \mathrm{u}(1)$ if $n = 1, 0$
18.	E_6	$\supset \mathrm{su}(2)_1 \oplus \mathrm{su}(2)_1$	$\oplus \mathrm{su}(4)_1$
19.	E_7	$\supset \mathrm{su}(2)_1 \oplus \mathrm{su}(2)_1$	$\oplus \mathrm{su}(2)_1 \oplus \mathrm{so}(8)_1$
20.	$\mathrm{sp}(3)$	$\supset \mathrm{su}(2)_1$	$\oplus \mathrm{sp}(2)_1$
21.	$\mathrm{sp}(5)$	$\supset \mathrm{su}(2)_1 \oplus \mathrm{su}(2)_1$	$\oplus \mathrm{sp}(3)_1$
22.	E_6	$\supset \mathrm{su}(3)_2 \oplus \mathrm{su}(2)_1$	$\oplus \mathrm{su}(2)_3$
23.	E_6	$\supset \mathrm{su}(3)_2 \oplus \mathrm{su}(2)_1$	$\oplus \mathrm{u}(1)$
24.	E_6	$\supset \mathrm{su}(3)_1 \oplus \mathrm{su}(2)_1$	$\oplus \mathrm{su}(3)_1 \oplus \mathrm{u}(1)$
25.	E_6	$\supset \mathrm{su}(2)_1 \oplus \mathrm{sp}(2)_2$	$\oplus \mathrm{u}(1)$
	$\mathrm{sp}(3) \oplus \underline{\mathrm{su}(2)}$	$\supset \mathrm{su}(3)_2 \oplus \underline{\mathrm{su}(2)_1}$	$\oplus \mathrm{u}(1)$
26.	E_6	$\supset \mathrm{su}(2)_1 \oplus \mathrm{sp}(2)_1$	$\oplus \mathrm{su}(2)_1 \oplus \mathrm{u}(1)$
	E_7	$\supset \mathrm{su}(3)_1 \oplus \mathrm{su}(2)_1$	$\oplus \mathrm{su}(4)_1 \oplus \mathrm{u}(1)$
27.	$\mathrm{sp}(3) \oplus \underline{\mathrm{su}(2)}$	$\supset \underline{\mathrm{su}(2)_1} \oplus \mathrm{sp}(2)_1$	$\oplus \mathrm{su}(2)_1$
	E_7	$\supset \mathrm{sp}(2)_1 \oplus \mathrm{su}(2)_1$	$\oplus \mathrm{so}(7)_1$
28.	E_7	$\supset \mathrm{sp}(2)_1 \oplus \mathrm{su}(2)_1$	$\oplus \mathrm{su}(2)_1 \oplus \mathrm{su}(2)_1 \oplus \mathrm{su}(2)_2$
29.	E_7	$\supset (G_2)_1 \oplus \mathrm{su}(2)_1$	$\oplus \mathrm{sp}(2)_1$
	$\mathrm{sp}(5)$	$\supset \mathrm{sp}(2)_1 \oplus \mathrm{su}(2)_1$	$\oplus \mathrm{sp}(2)_1$

Table 4: Algebra embeddings. The subscript on each subalgebra is its index of embedding.

[3] J. A. Minahan and D. Nemeschansky, *Nucl. Phys.* **B 482** (1996) 142 [`hep-th/9608047`]; *Nucl. Phys.* **B 489** (1997) 24 [`hep-th/9610076`].

[4] K. Kodaira, Ann. of Math. **77** (1963) 563; **78** (1963) 1.

[5] O. J. Ganor, *Nucl. Phys.* **B 488** (1997) 223 [`hep-th/9608109`].
    N. Seiberg, *Phys. Lett.* **B 388** (1996) 753 [`hep-th/9608111`].
    T. Banks, M. R. Douglas and N. Seiberg, *Phys. Lett.* **B 387** (1996) 278 [`hep-th/9605199`].
    K. Dasgupta and S. Mukhi, *Phys. Lett.* **B 385** (1996) 125 [`hep-th/9606044`].

[6] O. Aharony and Y. Tachikawa, "A holographic computation of the central charges of d=4, N=2 SCFTs," arXiv:0711.4532 [hep-th].

[7] E. Witten, *Phys. Lett.* **B 117** (1982) 324.

[8] P. C. Argyres, M. Crescimanno, A. D. Shapere and J. R. Wittig, "Classification of N = 2 superconformal field theories with two-dimensional Coulomb branches," [`hep-th/0504070`].

[9] N. Seiberg and E. Witten, *Nucl. Phys.* **B 431** (1994) 484 [`hep-th/9408099`].

[10] W. G. McKay and J. Patera, *Tables of Dimensions, Indices, and Branching Rules for Representations of Simple Lie Algebras*, Marcel Dekker (1981).

[11] P. C. Argyres and A. D. Shapere, *Nucl. Phys.* **B 461** (1996) 437 [`hep-th/9509175`].

[12] K. Landsteiner, E. Lopez, and D. Lowe, *J. High Energy Phys.* **9807** (1998) 011 [`hep-th/9805158`].

[13] P. C. Argyres and J. R. Wittig, "Classification of N = 2 superconformal field theories with two-dimensional Coulomb branches. II," [`hep-th/0510226`].

[14] S. Kass, R. V. Moody, J. Patera and R. Slansky, *Affine Lie Algebras, Weight Multiplicities, and Branching Rules*, volume 1, Univ. of California Press (1990).

[15] E. Witten, *Nucl. Phys.* **B 500** (1997) 3 [`hep-th/9703166`].